International Baccalaureate

# Extended essay cover

| *Candidates must complete this page and then give this cover and their final version of the extended essay to their supervisor.* | | | |
|---|---|---|---|
| Candidate session number | | | |
| Candidate name | | | |
| School number | | | |
| School name | | | |
| Examination session (May or November) | May | Year | 2013 |

Diploma Programme subject in which this extended essay is registered: Mathematics

(For an extended essay in the area of languages, state the language and whether it is group 1 or group 2.)

Title of the extended essay: How Fractals Reveal Our Misconceptions of Distance and Length in the Real World

## Candidate's declaration

*This declaration must be signed by the candidate; otherwise a grade may not be issued.*

The extended essay I am submitting is my own work (apart from guidance allowed by the International Baccalaureate).

I have acknowledged each use of the words, graphics or ideas of another person, whether written, oral or visual.

I am aware that the word limit for all extended essays is 4000 words and that examiners are not required to read beyond this limit.

This is the final version of my extended essay.

Candidate's signature: Date:

## Supervisor's report and declaration

*The supervisor must complete this report, sign the declaration and then give the final version of the extended essay, with this cover attached, to the Diploma Programme coordinator.*

Name of supervisor (CAPITAL letters)

*Please comment, as appropriate, on the candidate's performance, the context in which the candidate undertook the research for the extended essay, any difficulties encountered and how these were overcome (see page 13 of the extended essay guide). The concluding interview (viva voce) may provide useful information. These comments can help the examiner award a level for criterion K (holistic judgment). Do not comment on any adverse personal circumstances that may have affected the candidate. If the amount of time spent with the candidate was zero, you must explain this, in particular how it was then possible to authenticate the essay as the candidate's own work. You may attach an additional sheet if there is insufficient space here.*

TOLD ME THAT HE FIRST BECAME INTERESTED IN FRACTALS WHEN HIS DAD BOUGHT HIM A BOOK ABOUT FRACTALS IN THE 8th GRADE.

HAS BEEN A STRONG COMPETITOR IN MATH COMPETITIONS AND IS TAKING MATH HL THIS YEAR. HE STARTED HIS E.E. WITH A BROADER TOPIC AND FOCUSED TO THIS TITLE AFTER A FEW MONTHS. I SAW A ROUGHER COPY WITH LESS ILLUSTRATIONS A FEW MONTHS AGO. I MADE SOME COMMENTS ABOUT IMAGES THAT I BELIEVE HELPED HIM IMPROVE HIS FINAL COPY.

*This declaration must be signed by the supervisor; otherwise a grade may not be issued.*

I have read the final version of the extended essay that will be submitted to the examiner.

To the best of my knowledge, the extended essay is the authentic work of the candidate.

I spent 1.5 hours with the candidate discussing the progress of the extended essay.

Supervisor's signature

Date:

## Assessment form (for examiner use only)

| Criteria | Examiner 1 | maximum | Examiner 2 | maximum | Examiner 3 |
|---|---|---|---|---|---|
| | **Achievement level** | | | | |
| **A** research question | 1 | 2 | | 2 | |
| **B** introduction | 1 | 2 | | 2 | |
| **C** investigation | 2 | 4 | | 4 | |
| **D** knowledge and understanding | 2 | 4 | | 4 | |
| **E** reasoned argument | 2 | 4 | | 4 | |
| **F** analysis and evaluation | 2 | 4 | | 4 | |
| **G** use of subject language | 2 | 4 | | 4 | |
| **H** conclusion | 1 | 2 | | 2 | |
| **I** formal presentation | 2 | 4 | | 4 | |
| **J** abstract | 1 | 2 | | 2 | |
| **K** holistic judgment | 2 | 4 | | 4 | |
| Total out of 36 | 18 | | | | |

Extended Essay in Mathematics

# The Illusion of Length:

# How Fractals Reveal Our Misconceptions of Distance and Length in the Real World

May 2013

Word Count: 2966

Abstract:

Mathematics is a system used to help us represent and model the world around us so we can explain the complexity of nature by quantifying it into terms we can understand. However, this has certain limitations, and the subject area of fractal geometry shattered the neat and easy idea of dimensional geometry. I try to tackle the concepts of length and distance, and I try to describe why the two are not interchangeable. I will examine my question "Why is it that walking the length of a hypotenuse is shorter than walking the distance of the legs?" with relation to the famous question posed by mathematicians "How long is the British Coastline?" After extensive research into the subject field of Fractal Geometry, I present a brief history and the most basic concept of fractal geometry. I arrive at two conclusions. First, the concept of length and distance are not the same, and should never be confused. Length in the real world is not easily and possibly impossible to quantify, thus is irrelevant in most cases. Thus, we have an "illusion of length" when in most cases it cannot be applied to the world we live in. I also conclude that a human's walk is most likely not fractal, but is still very hard to quantify. I end with a thought experiment that could actually calculate how efficiently a human walks.

Word count: 230

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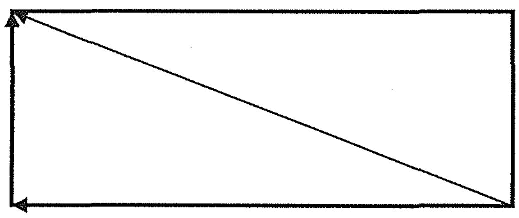
Introduction:

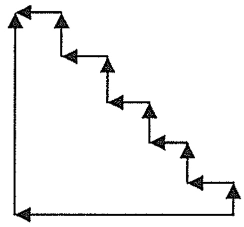
What is distance? According to the Merriam-Webster dictionary, the technical definition is "the degree or amount of separation between two points, lines, surfaces, or objects". I believe this definition is a very accurate description of the concept, but it is missing a pretty big point. We use this definition every day of our lives. Whether we are planning a trip or going to the local
grocery store, we use the concept of distance to describe how much we have to travel. However, when we say it is 5 miles to the grocery store or it is 300 miles to the nearest beach, those numbers are far from accurate. We are assuming that we are traveling from a specific start point to a specific end point, and more importantly, we are assuming we are traveling in a perfect 2 dimensional universe. Since the world we live in is chaotic and definitely not perfect, we cannot say we are traveling in a perfect 2 dimensional system, and thus cannot say it is 5 miles to the grocery store or 300 miles to the closest beach. For the same reason, it is impossible to answer the famous question: "How long is the coastline of Britain?"

I wrote this paper to look deeper into this concept, and answer the essential question in my head.
My research question is "How does Fractal Theory help us understand the concept of distance and length?"

Investigation:

Ever since I was a kid, one problem has always been in the back of my mind. When my friends and I walked home from school, we would always have to walk from one end to the other of a big soccer field. I always cut across the field until one day, one of my friends walked along the sides instead. I ended up reaching the end much faster than my friend, but I always wondered, we are traveling from the same point A to the same point B. Why is it that I am able to reach point B so much faster? The basic answer I learned from elementary school was that we both traveled from point A to point B. However, my friend traveled to point C, forming a right triangle. By definition, the two legs of a triangle sum up to be more than the hypotenuse. Thus, I am traveling less distance, explaining why I arrive faster than my friend. I was satisfied with this answer for a few years. Then I realized that if we very closely examine my path, I would be traveling one length of a soccer field and one width of a soccer field.





As seen in the diagram, my path seems to be a straight line. However, it can be made of the stair-like figure to the right on a miniscule scale. Thus, I am traveling just as much vertically and horizontally as my friend. I am in essence traveling the same path as my friend. So how is it that I am traveling faster than him?

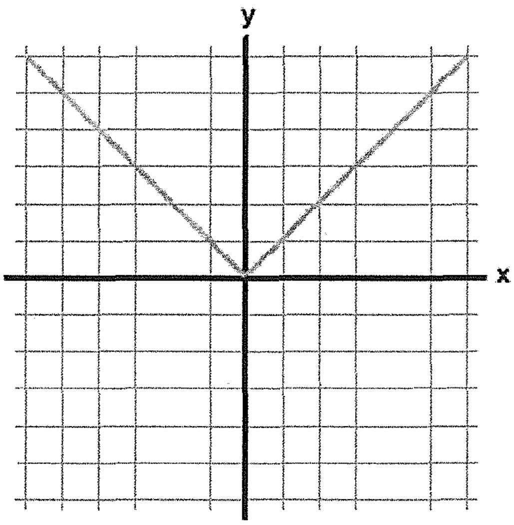
To answer this question, we must examine thoroughly the concept of Fractal Geometry and

the concept of length and dimensions.

The idea of fractals was first discovered by the Greek mathematician Archimedes, who envisioned the path of an arrow before it hits the target. He conjectured that before the arrow reaches the target, it has to at some point reach a halfway point. Before the arrow reaches the halfway point, it has to reach a point in the middle of the start and the halfway point, and so on ad infinitum. Archimedes theorized that no matter how small the length of the flight, it could always be cut in half.

Interestingly, the world of mathematics had not developed the field of fractal geometry until much later in the $19^{th}$ century. This can be attributed to the fact that the concept of fractal geometry seemed foreign but more importantly, useless. Most mathematics was focused on modeling the imperfect world in a calculable mathematical world. Fractals focused on describing the world as it was, and was therefore untouched by most mathematicians.

In 1872, Karl Weierstrass discovered the first fractal equation. To understand the equation Weierstrass presented, we first must explore the concept of a function that is continuous at a point but not differentiable. The following is a graph of the simple equation y= IxI,



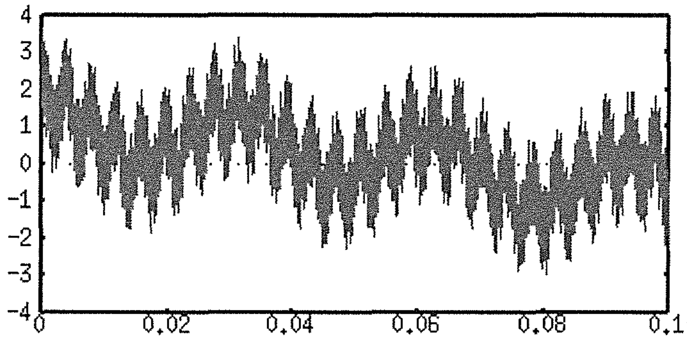
[1]

Notice that at x=0, the curve is continuous and by definition the function is continuous at the point x=0. However, the function is not differentiable at such point, as there is no line that is tangent to the point.

Weierstrass described a function that exhibited this behavior for all numbers in the domain. Such a theory was regarded as "too strange" for the world of mathematics, and was referred to the "monsters of mathematics. Weierstrass achieved the function by creating a series of infinite sums of cosine sequences so that there was every point on the graph was a vertex, and thus continuous but not differentiable at any point. The simplest representation of his function is shown below.

$$C(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$$

[1] Image Courtesy of Sparknotes.com. <http://img.sparknotes.com/figures/1/15debaf09555bfc7c688d9ee8ae574bc/abs_graph.gif>



[2]

When the function approaches a sum where n is a very large number, every point in the domain of the function will be a vertex point. David Hilbert took this concept and theorized that when n approaches infinity, there is no space between each vertex, thus the curve actually takes up space. After further research, Hilbert devised his own set that showed he could cover two dimensional areas with one dimensional lines of no area. This concept became central to fractal theory, as Hilbert described a way to transcend the first dimension into the second.

Soon, mathematicians realized that we can actually deal with fractional dimensions. Cantor, Koch, and Mandelbrot were mathematicians who really pioneered this study. Georg Cantor, who was actually Weierstrass's student, discovered the Cantor set 10 years after Weierstrass's fractal equation. This set takes a line segment as the starting condition. Then it takes the middle third out, leaving two line segments. It then repeats the process for the two line segments and so on.

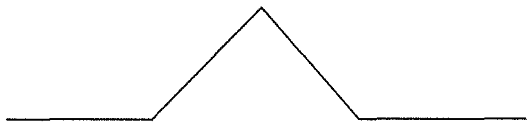
In 1904, Helge von Koch, a German mathematician took the concept of the Cantor set and

[2] Graph obtained from "plus.maths.orgjcontentjorigins-fractals"Graphical display shows a=8 and b=0.9

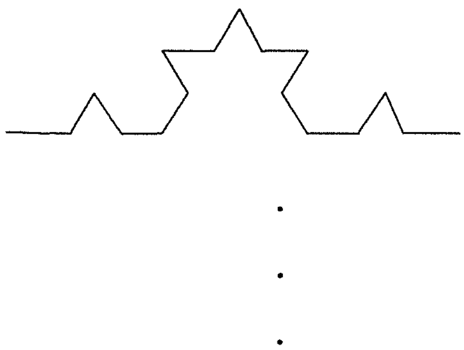
created Koch's curve. This curve is displayed below.



Stage 1



Stage 2



If we continue the trend for a large number of iterations, we can see that any part of the curve will look like the curve as a whole. Koch's concept became well known because it was the first of its kind; a fully visual fractal curve people can see.

The next development in fractals came from Benoit Mandelbrot. He was a brilliant man who took a closer examination at the famous question of the British Coastline, and then published a paper on the topic. Mandelbrot was the one who started to call the subject "fractals". His contribution to the subject was to give a concrete definition of fractal geometry, and to bring to light a subject that even mathematicians deemed too strange.

Unlike his predecessors, Mandelbrot had access to modem technology that could draw Koch's

curve to very high numbers of iterations. He could show the world fractals as people have never seen before. My favorite quote was from his famous book, The Fractal Geometry of Nature, which goes: "A cloud is made of billows upon billows upon billows that look like clouds. As you come closer to a cloud you don't get something smooth, but irregularities at a smaller scale."

Mandelbrot was a pioneer. He showed people that their conventions of mathematics and their applications in the real world were false. He showed that fractals cannot be expressed in whole number dimensions but rather needed to be expressed in fractional dimensions.

Lastly, the most important thing we need to understand about fractals is that they are chaotic. Let us take the example of Koch's curve, as illustrated on the previous page.

We need a basic understanding of chaos theory to state that the curve or any fractal curve is chaotic. To the simple eye, the diagram above illustrates a very simple and ordered sequence. A chaotic system can be identified through satisfaction of three conditions. First, a dynamic system can be defined as "chaotic" if it is sensitive to initial conditions. Second, the system must be topologically mixing. Third, the system's periodic orbits must be dense.[3]

A fractal curve or equation by definition is composed of many smaller versions of itself. Thus, the larger curve must reflect the shape of the smaller, initial portions. The initial portions affect what the fractal curve looks like, therefore satisfying chaos condition number 1. The concept of

[3] Antole Katok, Boris Hasselblatt, *A First Course in Dynamics: With a Panorama of Recent Developments*, Cambridge University Press, Cambridge, UK, 2003, Pages 205-219

topologically mixing is complex, and we will approach it from both a mathematical position and a generalist position. Generally, a system is said to be topologically mixing if the system "evolves" with iterations. Eventually, after a large number of iterations, no two sets of the system will be mutually exclusive. We can say that since a fractal is representative of itself, and "evolves" with each iteration, the fractal is topologically mixing. Approaching it from a mathematical position,

Lastly, we need to prove that fractals periodic orbits are dense. Basically what this means is every point in space is approached arbitrarily close by the system periodically. A fractal curve is capable of eventually covering any point in space. Wierstrass's function showed us that fractal functions may actually cover space on a plane. Thus, we can assume that the periodic orbits are dense, satisfying the third condition.

Now that I have given basis that fractal curves and equations are in fact chaotic, I can make the most important conjecture in this entire essay. Because fractal curves and systems are in fact chaotic, they are a fair representation of the real world. This is immensely important as it becomes the basis for my main argument point. The mathematic systems we use on a day-to-day basis for estimating distances and lengths are not chaotic. The two dimensional system is not chaotic, thus it is not a fair representation of the world we live in. This is why we cannot measure the coastline of England, and this is why we cannot measure any distance with a definite number.

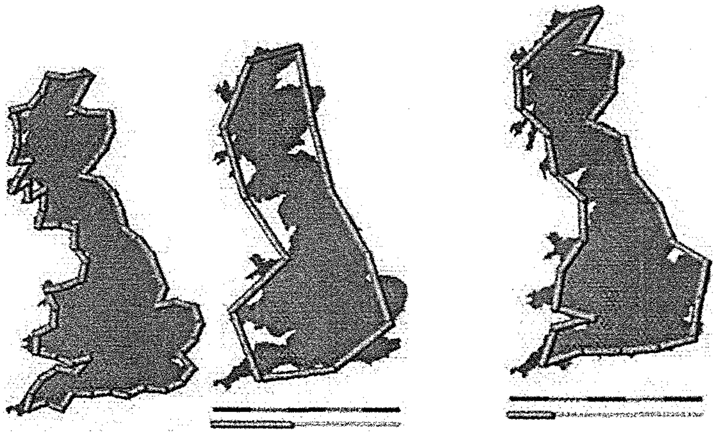
Let us take a break from discussion of fractals to discuss the mathematical system we are more

familiar with. We currently know of 4 dimensions with possibly a 5th dimension. We know length, width, and depth describe space, and the fourth dimension is time. When we deal with a distance, we are dealing strictly with the first two dimensions or three dimensions. Sometimes when we are dealing with large distances, we must take into account the curvature of the Earth, hence the third dimension. However, most of the time, when we are dealing with travel, we talk about distances in strictly two dimensions. We calculate the distance strictly as the number of miles it takes to travel to the destination, and we say with very high confidence that it takes so-and-so miles to travel to point B from point A.

If this was the case, why are we still not capable of measuring the length of the British coastline? It seems that mechanics are just not that simple on the real world. We could estimate the length by using a map and measuring out the length. However, we would soon discover that the length increases as the size of the ruler decreases. The problem that boggled British minds was eventually solved by Mandelbrot, who ventured that the coastline of Britain was not in any whole number dimension, thus we cannot measure it. If we were to zoom in at any point in the coastline, the image would be representative of the coastline as a whole. Thus, using an infinitely small ruler would result in a coastline of infinite length.

Mandelbrot addressed this phenomenon by stating "Clouds are not spheres, mountains are not cones, and lightening does not travel in a straight line. The complexity of nature's shapes differs in kind, not merely degree, from that of the shapes of ordinary geometry, the geometry of fractal shapes." His idea was that any portion of the coastline at any scale looks likes the coastline as a whole. He discovered the smaller the length of measure used to quantify the length of the

coastline, the larger the quantity turned out to be. His concept is illustrated below.



As we see, the length of the coastline seems to increase with each progression. Mandelbrot deduced that since the coast is fractal, the coastline is of infinite length and cannot be quantified with our current system.

Thus, is there ever such a thing as "length" in the true world? Can we ever quantify any distance in actuality? Can we say with certainty that anything is ever so-and-so miles away from something else?

By the Webster definition of distance, it can still exist with certainty. Distance is the amount of space separating two points, so we can measure this with a non-fractal curve. Since we are tracing the path between two points, it is possible to state the absolute distance between them. However, when we are referring to two points on the globe, we do take into account the curvature of the Earth.

However, the idea of length must be changed in order to show the existence of fractals in nature. We cannot say that a tree is 9 feet long, but we can say that the tree is 9 feet tall. In nature we can describe distance, but we cannot describe length. The problem is that our current math system uses length and distance interchangeably.

Imagine it this way; suppose I needed to walk across a room. While I need to travel 20 meters total, I need to avoid the couch and the TV, so I actually travel 30 meters to reach my destination. The 20 meters in the distance from start to finish, and is the distance. While the 30 meters, which is what I actually travel, is the length.

My central theory is that length cannot be quantified in nature due to fractals.

Let us apply all these principles to my original question, and once and for all solve my problem.

Since the path of the human stride is natural and chaotic, the path of the human stride is fractal. I will define the stride as the path one point in my body takes. The stride will be different for different points along my body. But since they all describe natural motions, they are all fractal. Thus the length of the stride is infinite.

There are two logical conclusion I can come up with is that the speed at which that point in my body travels is infinite. Assuming relativity can be ignored, or was already factored in, if we use smaller and more precise instruments, the path the point in my body travels will be longer and longer.

That conclusion seems less than possible, and would bend the laws of nature and time. Thus, Occam's razor suggests that the more logical conclusion is that the natural stride of man is not fractal.

In the context of my problem, although there is no way to tell exactly how much my body travels to get to the destination. If my path is in fact fractal, then any point on my body travels an infinite length. More likely, if it is not, my body still travels much more than the distance from start to finish, and it would be very hard to calculate exactly how much my body travels. What I failed to see was that my friend also travels in a natural human stride like me. Also, we defined speed as distance over time, not length traveled over time. Since we cover different distances, I was bound to reach the end faster. If we defined going the same speed as length

over time, the distance we traveled would depend on whose walk was more fractal by nature.

However, assuming he was a perfect robot and traveled exactly along a 2 dimensional line, and we traveled at the same rate of length over time, the result would be different. Since it is very difficult to measure how fractal the human stride is, and how much length I cover per minute, this can only be a thought exercise.

Conclusion:

The subject of fractals is incredibly interesting, as this was the first system in mathematics that directly describes a phenomenon in the real world. But it was a strange and new concept that mathematicians wouldn't dare touch the subject for centuries. What I did in this paper was to clarify the concept of fractals to solve a problem that has been bugging me. I tried to describe the path a human takes as fractal. I found that if it was in fact, completely fractal, then we would need to redefine some laws in physics. Since I have neither the time nor resources to destroy Einstein's theory, I arrived at the conclusion that the path a human takes is not fully fractal.

This can be used to solve my original problem through a thought experiment. In order to find my true speed, which is length traveled over time, extensive calculation and research must be performed. I can then compare it to my realistic speed, or distance traveled over time to calculate my walking efficiency. I don't have the resources to complete such an experiment.

I learned that fractals give us a new sense of the words length and distance. In the perfect world of mathematics, the two can be used interchangeably. But in the real world, with the existence of fractal geometry, length and distance become different. Length is infinite, but distance can be quantified. In the real world, we are under this illusion that length can be quantified, when it cannot.

Through our examination of the history and nature of fractals, we found the need to redefine the concepts of distance and length.

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